Scalar and fermionic vacuum currents in de Sitter spacetime with compact dimensions

S. Bellucci[1]*, A. A. Saharian[2]†, H. A. Nersisyan[2]

[1] *INFN, Laboratori Nazionali di Frascati, Via Enrico Fermi 40, 00044 Frascati, Italy*

[2] *Department of Physics, Yerevan State University, 1 Alex Manoogian Street, 0025 Yerevan, Armenia*

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Abstract

Vacuum expectation values (VEVs) of the current densities for charged scalar and Dirac spinor fields are investigated in $(D+1)$-dimensional de Sitter (dS) spacetime with toroidally compactified spatial dimensions. Along compact dimensions we impose quasiperiodicity conditions with arbitrary phases. In addition, the presence of a classical constant gauge field is assumed. The VEVs of the charge density and of the components for the current density along noncompact dimensions vanish. The gauge field leads to Aharonov-Bohm-like oscillations of the components along compact dimensions as functions of the magnetic flux. For small values of the comoving length of a compact dimension, compared with the dS curvature scale, the current density is related to the corresponding current in the Minkowski spacetime by a conformal relation. For large values of the comoving length and for a scalar field, depending on the mass of the field, two different regimes are realized with the monotonic and oscillatory damping of the current density. For a massive spinor field, the decay of the current density is always oscillatory. In supersymmetric models on the background of Minkowski spacetime with equal number of scalar and fermionic degrees of freedom and with the same phases in the periodicity conditions, the total current density vanishes due to the cancellation between the scalar and fermionic parts. The background gravitational field modifies the current densities for scalar and fermionic fields in different ways and for massive fields there is no cancellation in the dS spacetime.

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1 Introduction

The investigation of quantum effects in fixed gravitational backgrounds is among the most interesting topics in quantum field theory (for reviews see [1]). These effects may have important implications in black hole physics and in cosmology. The presence of the gravitational field, in general, reduces the number of symmetries and exact results can be obtained for highly symmetric backgrounds only. A better understanding of physical effects in these backgrounds could

*E-mail: bellucci@lnf.infn.it
†E-mail: saharian@ysu.am

serve as a handle to deal with more complicated geometries. In this context, the investigation of quantum field theoretical effects in de Sitter (dS) spacetime is of special interest. In inflationary scenario an approximately dS spacetime is employed to solve the problem of initial conditions in standard cosmology [2]. During an inflationary epoch in the early Universe the quantum fluctuations play a crucial role in the generation of cosmic structures from inflation. More recently observations of high-redshift Type Ia supernovae, galaxy clusters and cosmic microwave background [3] indicate that at the present epoch the Universe is accelerating and is well approximated by a model with a positive cosmological constant as a dominant source for the expansion. If the Universe would accelerate indefinitely, the standard cosmology would lead to an asymptotic dS universe. Therefore, the investigation of physical effects in dS spacetime is important for understanding both the early Universe and its future. An interesting topic which has received increasing attention is related to string-theoretical models of dS spacetime and inflation. A number of constructions of metastable dS vacua were discussed within the framework of string theory (see, for instance, [4]).

In recent years much attention has been paid to the possibility for the universe to have non-trivial topology. In particular, a number of fundamental physical theories are formulated in spacetimes with compact extra dimensions. The idea of compactified dimensions has been extensively used in supergravity and superstring theories. The models of a compact universe with non-trivial topology may play an important role by providing proper initial conditions for inflation [5]. The quantum creation of the universe having toroidal spatial topology is discussed in [6] and in Refs. [7] within the framework of supergravity theories. As it was argued in Refs. [8], there is no reason to believe that the version of dS spacetime which may emerge from string theory, will necessarily be the most familiar version with symmetry group $O(1,4)$ and there are many different topological spaces which can accept the dS metric locally. There are several reasons to expect that in string theory the most natural topology for the universe is that of a flat compact three-manifold.

The non-trivial topology of the background space can have important physical implications in quantum field theory. The periodicity conditions imposed on fields along compact dimensions give rise to the modification of the spectrum for normal modes and, related to this, the expectation values of physical observables are changed. A well known effect of this type, demonstrating the connection between quantum phenomena and global properties of spacetime, is the topological Casimir effect [9, 10]. The Casimir energy of bulk fields induces a non-trivial potential for the compactification radius providing a stabilization mechanism for moduli fields and thereby fixing the effective gauge couplings. The Casimir effect has also been considered as a possible origin for the dark energy in both Kaluza-Klein-type and braneworld models [11].

One-loop quantum effects for scalar and Dirac spinor fields induced by the toroidal compactification of spatial dimensions in dS spacetime have been recently investigated in Refs. [12, 13, 14, 15]. In these papers the vacuum expectation values (VEVs) for the field squared and the energy-momentum tensor are considered for untwisted and twisted fields. These quantities are among the most important local characteristics of the quantum vacuum and are closely related with the structure of spacetime through the theory of gravitation. For charged fields another important bilinear characteristic is the VEV of the current density. In addition to describing the physical structure of the quantum field at a given point, the current acts as the source in the Maxwell equations and plays an important role in modeling a self-consistent dynamics involving the electromagnetic field.

In the present paper, we consider the combined effects of topology and the gravitational field on the VEVs of the current density for charged scalar and fermionic fields in the background of dS spacetime with an arbitrary number of toroidally compactified spatial dimensions. Along compact dimensions we impose generic quasiperiodic boundary conditions with arbitrary phases

which, as special cases, include the periodicity conditions for untwisted and twisted fields. In addition, the presence of a classical constant gauge field will be assumed. As it will be seen, this leads to Aharonov-Bohm-like effects on the VEV of the current density. The paper is organized as follows. In the next section we describe the background geometry and evaluate the Hadamard function for a complex scalar field assuming that the field is prepared in the Bunch-Davies vacuum state. By using this function, in section 3, we investigate the VEV of the current density. The current density for the Dirac spinor field is discussed in section 4. The main results are summarized in section 5.

2 Hadamard function for a scalar field

In the present paper the background geometry is described by the $(D+1)$-dimensional dS line-element in planar coordinates:

$$ds^2 = dt^2 - e^{2t/\alpha} \sum_{l=1}^{D} (dx^l)^2. \tag{1}$$

The corresponding metric tensor is the maximally symmetric vacuum solution of Einstein equations with the cosmological constant $\Lambda = D(D-1)\alpha^{-2}/2$. For the coordinates $\mathbf{x}_p = (x^1, ..., x^p)$ one has $-\infty < x^l < \infty$, $l = 1, \ldots, p$, and we assume that the coordinates $\mathbf{x}_q = (x^{p+1}, ..., x^D)$, with $q = D - p$, are compactified to circles with the lengths L_l: $0 \leq x^l \leq L_l$, $l = p+1, \ldots, D$. Hence, we consider the spatial topology $R^p \times (S^1)^q$. In the discussion below it will be convenient, in addition to the time coordinate t, to use the conformal time $\tau = -\alpha e^{-t/\alpha}$ with $-\infty < \tau < 0$. For this coordinate the line element takes a conformally flat form with the conformal factor $(\alpha/\tau)^2$: $ds^2 = (\alpha/\tau)^2 \eta_{\mu\nu} dx^\mu dx^\nu$, where $\eta_{\mu\nu}$ is the Minkowskian metric tensor.

Firstly, we consider a complex scalar field $\varphi(x)$ with a curvature coupling parameter ξ, in the presence of a classical abelian gauge field A_μ. The corresponding field equation has the form

$$\left(D_\mu D^\mu + m^2 + \xi R\right) \varphi(x) = 0, \tag{2}$$

where $D_\mu = \nabla_\mu + ieA_\mu$ is the gauge-covariant derivative and e is the coupling between the scalar and gauge fields. For the scalar curvature in (2) one has $R = D(D+1)/\alpha^2$. In the most important special cases of minimally and conformally coupled scalars one has $\xi = 0$ and $\xi = (D-1)/(4D)$, respectively. Since the background space is multiply-connected, in addition to the field equation one should specify the periodicity conditions along compact dimensions. We will assume generic quasiperiodic boundary condition

$$\varphi(t, \mathbf{x}_p, \mathbf{x}_q + L_l \mathbf{e}_l) = e^{i\alpha_l} \varphi(t, \mathbf{x}_p, \mathbf{x}_q), \tag{3}$$

with constant phases α_l and with $\mathbf{e}_l$ being the unit vector along the dimension x^l, $l = p+1, ..., D$ (for physical effects of phases in periodicity conditions for fields in multiply-connected spaces see also [16]). The results below will be periodic functions of α_l with the period equal to 2π. The special cases of untwisted and twisted scalar fields (periodic and antiperiodic boundary conditions), most frequently discussed in the literature, correspond to $\alpha_l = 0$ and $\alpha_l = \pi$, respectively.

Here we will consider the simplest configuration of the gauge field with $A_\mu = \text{const}$. In this case, the gauge field can be excluded from the field equation by the gauge transformation

$$A'_\mu = A_\mu + \partial_\mu \omega, \ \varphi'(x) = e^{-ie\omega} \varphi(x), \ \omega = -A_\mu x^\mu. \tag{4}$$

In the new gauge one has $A'_\mu = 0$ and $D'_\mu = \nabla_\mu$. Now the quasiperiodicity condition takes the form

$$\varphi'(t, \mathbf{x}_p, \mathbf{x}_q + L_l \mathbf{e}_l) = e^{i\tilde{\alpha}_l} \varphi'(t, \mathbf{x}_p, \mathbf{x}_q), \tag{5}$$

where

$$\tilde{\alpha}_l = \alpha_l + eA_l L_l. \tag{6}$$

As it is seen, the phases in the periodicity conditions and the value of the gauge field are related to each other through a gauge transformation. In what follows, the physical results will depend on the phases in the periodicity conditions and on the gauge potential in the form of the combination (6). Note that if $\mathbf{A}$ is the spatial vector with D components corresponding to the spacetime vector A_μ, then $\mathbf{A}_l = -A_l$. Although the corresponding field strength vanishes, a constant gauge field shifts the phases in the periodicity conditions along compact dimensions. As it will be seen below, this leads to the Aharonov-Bohm-like effects on the current density for charged fields. Note that in (6) the shift due to the gauge field may be written in the form $eA_l L_l = -e\mathbf{A}_l L_l = -2\pi\Phi_l/\Phi_0$, where $\Phi_0 = 2\pi/e$ is the flux quantum and Φ_l is the flux enclosed by the circle corresponding to the lth compact dimension. In what follows we will work in terms of the gauge transformed field $\varphi'(x)$ omitting the prime.

The physical quantity we are interested in is the current density for a charged scalar field. The corresponding operator is given by the standard expression:

$$j_\mu(x) = ie[\varphi^+(x) D_\mu \varphi(x) - (D_\mu \varphi^+(x))\varphi(x)]. \tag{7}$$

Note that in the gauge under consideration one has $D_\mu \varphi = \nabla_\mu \varphi = \partial_\mu \varphi$. The VEV of the current density, $\langle j_\mu(x) \rangle = \langle 0 | j_\mu(x) | 0 \rangle$, is expressed in terms of the Hadamard function

$$G^{(1)}(x, x') = \langle 0 | \varphi(x)\varphi^+(x') + \varphi^+(x')\varphi(x) | 0 \rangle , \tag{8}$$

where $|0\rangle$ stands for the vacuum state. The corresponding formula reads:

$$\langle j_\mu(x) \rangle = \frac{i}{2} e \lim_{x' \to x} (\partial_\mu - \partial'_\mu) G^{(1)}(x, x'). \tag{9}$$

By expanding the field operator in terms of a complete set $\{\varphi_\sigma^{(\pm)}(x)\}$ of solutions to the field equation, obeying the periodicity conditions (5), and using the standard commutation relations for the annihilation and creation operators, the following formula is obtained for the Hadamard function:

$$G^{(1)}(x, x') = \sum_\sigma \sum_{s=\pm} \varphi_\sigma^{(s)}(x) \varphi_\sigma^{(s)*}(x'). \tag{10}$$

Here $\sum_\sigma$ includes the summation over the discrete components of the collective index σ and the integration over the continuous ones (for the specification of the collective index in the geometry under consideration see below).

Because of the plane symmetry of the problem, the dependence of the scalar mode functions on spatial coordinates can be taken in the standard exponential form, $e^{i\mathbf{k}\cdot\mathbf{x}}$, with $\mathbf{x} = (\mathbf{x}_p, \mathbf{x}_q)$ and $\mathbf{k} = (\mathbf{k}_p, \mathbf{k}_q)$. From the field equation it can be seen that the general solution for the time-dependent part of the mode functions is a linear combination of the functions $\eta^{D/2} H_\nu^{(1)}(k\eta)$ and $\eta^{D/2} H_\nu^{(2)}(k\eta)$, where $k = |\mathbf{k}|$,

$$\eta = |\tau| = \alpha e^{-t/\alpha}, \tag{11}$$

and the order of the Hankel functions $H_\nu^{(1,2)}(z)$ is related to the mass of the field by

$$\nu = \left[D^2/4 - D(D+1)\xi - m^2\alpha^2 \right]^{1/2}. \tag{12}$$

This parameter is either real and nonnegative or purely imaginary.

An important step in formulating quantum field theory in a curved spacetime is the choice of vacuum. Different choices of the coefficients in the linear combination of the Hankel functions correspond to different choices of the vacuum state. dS spacetime has maximal symmetry and it is natural to choose a vacuum state having the same symmetry. It is well known that in dS spacetime there exists a one-parameter family of maximally symmetric vacuum states, which have been dubbed the α-vacua (see Ref. [17] and references therein). Among them the Bunch-Davies vacuum [18] is the only one with the Hadamard singularity structure.

Here we assume that the scalar field is prepared in the Bunch-Davies vacuum. The mode functions realizing this state read:

$$\begin{aligned}\varphi_{\sigma}^{(+)}(x) &= C_{\sigma}^{(+)}\eta^{D/2}H_{\nu}^{(1)}(k\eta)e^{i\mathbf{k}_p\cdot\mathbf{x}_p+i\mathbf{k}_q\cdot\mathbf{x}_q},\\ \varphi_{\sigma}^{(-)}(x) &= C_{\sigma}^{(-)}\eta^{D/2}H_{\nu^*}^{(2)}(k\eta)e^{i\mathbf{k}_p\cdot\mathbf{x}_p+i\mathbf{k}_q\cdot\mathbf{x}_q}.\end{aligned} \tag{13}$$

In these expressions, for the components of the momentum along uncompactified dimensions one has $-\infty < k_l < +\infty$, $l = 1, \ldots, p$, and the eigenvalues of the components along the compact dimensions are quantized by the periodicity conditions (5):

$$k_l = (2\pi n_l + \tilde{\alpha}_l)/L_l, \quad n_l = 0, \pm1, \pm2, \ldots, \tag{14}$$

for $l = p+1, ..., D$. Hence, the mode functions are specified by the set of quantum numbers $\sigma = (\mathbf{k}_p, \mathbf{n}_q)$, where $\mathbf{n}_q = (n_{p+1}, \ldots, n_D)$.

The coefficients $C_{\sigma}^{(\pm)}$ in (13) are determined from the orthonormalization condition

$$\int d^D x\sqrt{|g|}\left[\varphi_{\sigma}^{(s)}(x)\partial_t\varphi_{\sigma'}^{(s')*}(x) - \varphi_{\sigma'}^{(s')*}(x)\partial_t\varphi_{\sigma}^{(s)}(x)\right] = i\delta_{\sigma\sigma'}\delta_{ss'}, \tag{15}$$

with $\delta_{\sigma\sigma'} = \delta(\mathbf{k}_p - \mathbf{k}'_p)\prod_{l=p+1}^{D}\delta_{n_l n'_l}$ and g being the determinant of the metric tensor corresponding to the line-element (1). By using the Wronskian relation for the Hankel functions, one finds

$$|C_{\sigma}^{(\pm)}|^2 = \frac{\alpha^{1-D}e^{i(\nu-\nu^*)\pi/2}}{2^{p+2}\pi^{p-1}V_q}, \tag{16}$$

where the star stands for the complex conjugate and $V_q = L_{p+1}...L_D$ is the volume of the compact subspace.

Substituting the functions (13) into the mode-sum formula (10), for the Hadamard function we get the representation

$$\begin{aligned}G^{(1)}(x,x') &= \frac{(\eta\eta')^{D/2}\,e^{i(\nu-\nu^*)\pi/2}}{2^{p+2}\pi^{p-1}V_q\alpha^{D-1}}\int d\mathbf{k}_p\,e^{i\mathbf{k}_p\cdot\Delta\mathbf{x}_p}\sum_{\mathbf{n}_q}e^{i\mathbf{k}_q\cdot\Delta\mathbf{x}_q}\\ &\quad\times[H_{\nu}^{(1)}(k\eta)H_{\nu^*}^{(2)}(k\eta') + H_{\nu^*}^{(2)}(k\eta)H_{\nu}^{(1)}(k\eta')],\end{aligned} \tag{17}$$

where $\Delta\mathbf{x}_p = \mathbf{x}_p - \mathbf{x}'_p$, $\Delta\mathbf{x}_q = \mathbf{x}_q - \mathbf{x}'_q$. In (17) and in the formulas below we use the notation

$$\sum_{\mathbf{n}} = \sum_{n_1=-\infty}^{+\infty}\cdots\sum_{n_l=-\infty}^{+\infty}, \tag{18}$$

for $\mathbf{n} = (n_1, \ldots, n_l)$. Note that for a scalar field in dS spacetime without compact dimensions ($p = D$), the corresponding two-point functions contain infrared divergences for $\mathrm{Re}\,\nu \geq D/2$ and in this region the Bunch-Davies vacuum is not a physically realizable state. In particular,

this is the case for a minimally coupled massless scalar field. These divergences come from the singular behavior of the Hankel functions at the origin. For the topology under consideration, assuming that $\sum_{l=p+1}^{D}\tilde{\alpha}_l^2 \neq 0$ and $|\tilde{\alpha}_l| < \pi$, the momentum k has a nonzero minimum value $k_{\min} = \sqrt{\sum_{l=p+1}^{D}\tilde{\alpha}_l^2/L_l^2}$. In this case the two-point function (17) contains no infrared divergences and the Bunch-Davies vacuum is a physically realizable state for all values of the parameter ν.

For the further evaluation of the Hadamard function, given by (17), we apply to the series over n_r, $p+1 \leq r \leq D$, the Abel-Plana-type summation formula [19, 20]

$$\frac{2\pi}{L_r}\sum_{n_r=-\infty}^{\infty} g(k_r)f(|k_r|) = \int_0^{\infty} dz[g(z)+g(-z)]f(z) + i\int_0^{\infty} dz\,[f(iz)-f(-iz)]\sum_{\lambda=\pm 1}\frac{g(i\lambda z)}{e^{zL_r+i\lambda\tilde{\alpha}_r}-1}, \tag{19}$$

with k_r defined by (14). In the special case $g(z)=1$ and $\tilde{\alpha}_r=0$ this formula is reduced to the standard Abel-Plana formula (for generalizations of the Abel-Plana formula see [21]). For the series in (17) one has $g(z)=e^{iz\Delta x^r}$ and the function $f(z)$ is given by the expression in the square brackets. It can be seen that the contribution of the first integral in the right-hand side of (19) to the Hadamard function coincides with the corresponding Hadamard function for the topology $R^{p+1}\times(S^1)^{q-1}$ with the lengths of the compact dimensions $(L_{p+1},\ldots,L_{r-1},L_{r+1},\ldots,L_D)$.

In the part corresponding to the contribution of the second integral in (19) we use the expansion $e^y-1=\sum_{n=1}^{\infty}e^{-ny}$. After some intermediate calculations, the Hadamard function is presented in the form

$$\begin{aligned} G^{(1)}(x,x') &= \frac{4\,(\eta\eta')^{D/2}L_r}{(2\pi)^{(p+3)/2}V_q\alpha^{D-1}}\sum_{\mathbf{n}_{q-1}^{(r)}} e^{i\mathbf{k}_{q-1}^{(r)}\cdot\Delta\mathbf{x}_{q-1}}\int_0^{\infty} dz\,z \\ &\times\left[I_{-\nu}(\eta z)K_\nu(\eta' z)+K_\nu(\eta z)I_\nu(\eta' z)\right]\sum_{n=-\infty}^{\infty} e^{-in\alpha_r} \\ &\times\frac{f_{(p-1)/2}(\sqrt{z^2+u_r^2}\sqrt{(\Delta x^r+nL_r)^2+|\Delta\mathbf{x}_p|^2})}{[(\Delta x^r+nL_r)^2+|\Delta\mathbf{x}_p|^2]^{(p-1)/2}}, \end{aligned} \tag{20}$$

where $I_\nu(y)$ and $K_\nu(y)$ are the modified Bessel functions, $\mathbf{n}_{q-1}^{(r)}=(n_{p+1},\ldots,n_{r-1},n_{r+1},\ldots,n_D)$, $\mathbf{k}_{q-1}^{(r)}=(k_{p+1},\ldots,k_{r-1},k_{r+1},\ldots,k_D)$ and

$$u_r^2 = |\mathbf{k}_{q-1}^{(r)}|^2 = \sum_{l=p+1,\neq r}^{D}(2\pi n_l+\tilde{\alpha}_l)^2/L_l^2. \tag{21}$$

Here and in what follows we use the notation

$$f_\mu(z) = z^\mu K_\mu(z). \tag{22}$$

The representation (20) of the Hadamard function is valid for $\mathrm{Re}\,\nu<1$. The $n=0$ term in this formula is the Hadamard function for the topology $R^{p+1}\times(S^1)^{q-1}$ with the lengths of the compact dimensions $(L_{p+1},\ldots,L_{r-1},L_{r+1},\ldots,L_D)$. The remaining part vanishes in the limit $L_r\to\infty$ and it is induced by the compactification of the rth dimension. The expression (20) can be used for the investigation of the VEVs for various physical observables. Here we consider the VEV of the current density.

3 Current density for a scalar field

Having the Hadamard function, we can evaluate the expectation value for the current density by using the formula (9). It can be seen that the VEVs of the charge density ($\mu = 0$) and of the components of the current density along uncompactified dimensions vanish:

$$\langle j_\mu \rangle = 0,\ \mu = 0, 1, \ldots, p. \tag{23}$$

The latter is a direct consequence of the problem symmetry. For a noncompact dimension x^l the problem is symmetric under the reflection $x^l \to -x^l$. The presence of the nonzero current density along x^l would break this symmetry.

For the VEV of the current density along the rth compact dimension, from (9) and (20) we find

$$\begin{aligned}\langle j^r \rangle &= \frac{8e\alpha(\eta/\alpha)^{D+2}}{(2\pi)^{(p+3)/2} V_q L_r^{p-1}} \int_0^\infty dz\, z \left[I_{-\nu}(\eta z) + I_\nu(\eta z)\right] K_\nu(\eta z) \\ &\quad \times \sum_{n=1}^{\infty} \frac{\sin(n\tilde{\alpha}_r)}{n^p} \sum_{\mathbf{n}_{q-1}^{(r)}} f_{(p+1)/2}(nL_r\sqrt{z^2 + u_r^2}).\end{aligned} \tag{24}$$

This VEV is an even periodic function of the phases $\tilde{\alpha}_l$, $l \neq r$, with the period 2π, and it is an odd periodic function of the phase $\tilde{\alpha}_r$ with the same period. In particular, the current density is a periodic function of the magnetic flux with the period of the flux quantum. In the absence of the gauge field the VEV of the current density vanishes for special cases of twisted ($\tilde{\alpha}_r = \pi$) and untwisted ($\tilde{\alpha}_r = 0$) fields. In the special case of a single compact dimension one has $p = D - 1$, $q = 1$, and the general formula takes the form

$$\langle j^r \rangle = \frac{8e\alpha(\eta/\alpha)^{D+2}}{(2\pi)^{D/2+1} L_r^{D-1}} \int_0^\infty dz\ z \left[I_{-\nu}(\eta z) + I_\nu(\eta z)\right] K_\nu(\eta z) \sum_{n=1}^{\infty} \frac{\sin(n\tilde{\alpha}_r)}{n^{D-1}} f_{D/2}(nL_r z). \tag{25}$$

From the covariant conservation equation $\nabla_\mu \langle j^\mu \rangle = 0$ it follows that the charge flux through the $(D-1)$-dimensional spatial hypersurface $x^r = \text{const}$ is determined by the quantity $n_r \langle j^r \rangle$, where $n_r = \sqrt{|g_{rr}|} = \alpha/\eta$ is the normal to the hypersurface. Now, from (24) we see that this quantity depends on the time coordinate and on the lengths of the compact dimensions in the form of the ratio L_l/η:

$$n_r \langle j^r \rangle = \alpha^{-D} f(L_{p+1}/\eta, \ldots, L_D/\eta). \tag{26}$$

By taking into account that $L_l^{(\mathrm{p})} = \alpha L_l/\eta$ is the proper length of the compact dimension, we see that the ratio L_l/η is the proper length of the lth compact dimension measured in the units of the dS curvature scale α.

For a conformally coupled massless scalar field one has $m = 0$, $\xi = (D-1)/(4D)$ and, hence, $\nu = 1/2$. By taking into account the corresponding expressions for the modified Bessel function, we see that $\left[I_{-\nu}(x) + I_\nu(x)\right] K_\nu(x) = 1/x$. The integral in (24) is evaluated by using the formula

$$\int_0^\infty du\, f_\mu(a\sqrt{u^2 + b^2}) = \frac{1}{a}\sqrt{\frac{\pi}{2}} f_{\mu+1/2}(ab), \tag{27}$$

and for the VEV of the current density we get

$$\langle j^r \rangle = \frac{4e(\eta/\alpha)^{D+1}}{(2\pi)^{p/2+1} V_q L_r^p} \sum_{n=1}^{\infty} \frac{\sin(n\tilde{\alpha}_r)}{n^{p+1}} \sum_{\mathbf{n}_{q-1}^{(r)}} f_{p/2+1}(nL_r u_r). \tag{28}$$

In this case, the current density is related to the corresponding result in the Minkowski spacetime (see Ref. [22]) with compact dimensions of the lengths $(L_{p+1}, \ldots, L_D)$ by the equation $\langle j^r \rangle = (\eta/\alpha)^{D+1} \langle j^r \rangle^{(\mathrm{M})}$.

Let us consider the Minkowskian limit for the general expression of the current density given by (24). This corresponds to the limit $\alpha \to \infty$ for a fixed value of t. In this limit one has $\nu \approx i\beta$, with $\beta = m\alpha \gg 1$ and $\eta \approx \alpha - t$. By using the uniform asymptotic expansions for the modified Bessel functions for imaginary values of the order with large modulus (see, for example, Ref. [23]), we can see that for $x < 1$ one has

$$[I_{i\beta}(\beta x) + I_{-i\beta}(\beta x)]\, K_{i\beta}(\beta x) \sim \frac{1}{\beta} \cos[2\beta f(x)], \tag{29}$$

where

$$f(x) = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}. \tag{30}$$

In the case $x > 1$ the leading term is given by

$$[I_{i\beta}(\beta x) + I_{-i\beta}(\beta x)]\, K_{i\beta}(\beta x) \sim \frac{1}{\beta\sqrt{x^2-1}}. \tag{31}$$

From these expressions it follows that the dominant contribution to the current density in (24) comes from the integration range $z > m$. In this range, by using (31) and the integration formula (27), to the leading order we find

$$\langle j^r \rangle \approx \frac{4eL_r^{-p}}{(2\pi)^{p/2+1} V_q} \sum_{n=1}^{\infty} \frac{\sin(n\tilde{\alpha}_r)}{n^{p+1}} \sum_{\mathbf{n}_{q-1}^{(r)}} f_{p/2+1}(nL_r\sqrt{u_r^2+m^2}). \tag{32}$$

The expression in the right-hand side coincides with the VEV of the current density in the Minkowski bulk with toroidally compactified dimensions [22]. For this background geometry, the finite temperature corrections to the expectation value of the current density have been investigated in Ref. [22]. In this reference, we have also derived an alternative expression for the current density in the Minowskian bulk by using the zeta function approach. In the case of a massless field, the expression in the right-hand side of (32) reduces to the result which follows from (28).

Now we turn to the investigation of the current density in the asymptotic regions of the ratio L_r/η. As we have mentioned before, this quantity is the ratio of the comoving length of the compact dimension to the curvature radius of dS spacetime. For small values of this ratio, $L_r/\eta \ll 1$, we introduce in (24) a new integration variable $y = L_r z$. By taking into account that for large values of x one has $[I_{-\nu}(x) + I_\nu(x)]\, K_\nu(x) \approx 1/x$, we find that to the leading order $\langle j^r \rangle$ coincides with the corresponding result for a conformally coupled massless field (see (28)):

$$\langle j^r \rangle \approx (\eta/\alpha)^{D+1} \langle j^r \rangle^{(\mathrm{M})},\ L_r/\eta \ll 1. \tag{33}$$

For a fixed value of the ratio L_r/α, the limit under consideration corresponds to the early stages of the cosmological expansion, $t \to -\infty$, and the current density behaves like $\exp[-(D+1)t/\alpha]$. Note that in the limit $L_r/\eta \ll 1$, the leading term in the VEV of the quantity $n_r \langle j^r \rangle$ coincides with the corresponding quantity in the Minkowskian bulk, if we replace the lengths of the compact dimensions L_l by the proper lengths $\alpha L_l/\eta$. In this limit, the effects induced by the curvature of the background spacetime are small.

In the limit of large values of the proper length of the rth compact dimension, compared with the dS curvature scale, one has $\eta/L_r \ll 1$. In this case, we introduce in (24) a new integration

variable $y = L_r z$ and then expand the integrand by using the formulas for the modified Bessel functions for small values of the argument. Two cases should be considered separately. For positive values of the parameter ν, after the integration over y by using the formula from [24], to the leading order we find

$$\langle j^r \rangle \approx e \frac{2^{\nu-(p-1)/2}(\eta/L_r)^{D-2\nu+2}\Gamma(\nu)}{\pi^{(p+3)/2}\alpha^{D+1}V_{q-1}L_r^{p-D}} \sum_{n=1}^{\infty} \frac{\sin(n\tilde{\alpha}_r)}{n^{p-2\nu+2}} \sum_{\mathbf{n}_{q-1}^{(r)}} f_{(p+3)/2-\nu}(nL_r u_r), \tag{34}$$

for $\eta/L_r \ll 1$. In the case of a conformally coupled massless scalar field $\nu = 1/2$ and (34) reduces to the exact result given by (28). For a fixed value of L_r/α, the limit under consideration corresponds to late stages of the cosmological evolution, $t \to +\infty$, and the current density is suppressed by the factor $\exp[-(D - 2\nu + 2)t/\alpha]$. Note that formula (34) also describes the asymptotic behavior for the current density in the strong curvature regime corresponding to small values of the parameter α. In the model with a single compact dimension ($p = D - 1$) the asymptotic formula (34) takes the form

$$\langle j^r \rangle \approx \frac{2e\Gamma(\nu)\Gamma(D/2 + 1 - \nu)\eta}{\pi^{D/2+1}\alpha^{D+1}(L_r/\eta)^{D-2\nu+1}} \sum_{n=1}^{\infty} \frac{\sin(n\tilde{\alpha}_r)}{n^{D-2\nu+1}}. \tag{35}$$

The sum in the latter expression is expressed in terms of the Hurwitz zeta function $\zeta(s, x)$.

In the same limit, $\eta/L_r \ll 1$, and for pure imaginary values of the parameter ν, by taking into account that for small values of x one has

$$[I_{-\nu}(x) + I_{\nu}(x)]\, K_{\nu}(x) \approx \mathrm{Re}\left[\frac{(2/x)^{2i|\nu|}\Gamma(i|\nu|)}{\Gamma(1 - i|\nu|)}\right], \tag{36}$$

for the current density we find the following asymptotic behavior

$$\langle j^r \rangle \approx \frac{8e\alpha B_s e^{-(D+2)t/\alpha}}{(2\pi)^{(p+3)/2}\, V_q L_r^{p+1}} \cos[2|\nu|t/\alpha + 2|\nu|\ln(L_r/\alpha) + \phi_s]. \tag{37}$$

In this formula, B_s and ϕ_s are defined by the relation

$$B_s e^{i\phi_s} = 2^{i|\nu|}\Gamma(i|\nu|) \sum_{n=1}^{\infty} \frac{\sin(n\tilde{\alpha}_r)}{n^{p+2-2i|\nu|}} \sum_{\mathbf{n}_{q-1}^{(r)}} f_{(p+3)/2-i|\nu|}(nL_r u_r). \tag{38}$$

Hence, in the case under consideration, at late stages of the cosmological evolution the current density is suppressed by the factor $e^{-(D+2)t/\alpha}$ and the damping of the corresponding VEV has an oscillatory nature.

For a fixed value of the time coordinate t, the condition $\eta/L_r \ll 1$ corresponds to the length of the rth compact dimension much larger than the dS curvature scale, $L_r \gg \alpha$. Now for the values of the curvature scale $\alpha \gtrsim m^{-1}$ one also has $L_r \gg m^{-1}$ and the limit we have discussed corresponds to the length of the compact dimension much larger than the Compton wavelength of the field quanta. As it is seen from the formula (35), in this range the decay of the current density is a power-law as a function of mL_r. This is in contrast to the case of the Minkowski bulk, where, as it can be seen from (32), the VEV of the current density is suppressed exponentially. For example, in the case of a single compact dimension ($q = 1$) one has

$$\langle j^r \rangle^{(\mathrm{M})} \approx \frac{2em^D \sin(\tilde{\alpha}_r)}{(2\pi)^{D/2}\,(mL_r)^{D/2}} e^{-mL_r},$$

for $mL_r \gg 1$.

In figure 1 we display the quantity $\alpha^D n_r \langle j^r \rangle /e$ as a function of the phase in the quasiperiodicity condition for a conformally coupled scalar field in the $D = 4$ Kaluza-Klein-type model with a single compact dimension $x^r = x^4$ ($p = 3$). The graphs are plotted for $\alpha m = 0.25$ and the numbers near the curves are the corresponding values for the ratio L_r/η. Let us recall that the current density is a periodic function of $\tilde{\alpha}_r$ with the period equal 2π. For a field with the periodic boundary condition ($\alpha_r = 0$) one has $\tilde{\alpha}_r/2\pi = -\Phi_r/\Phi_0$.

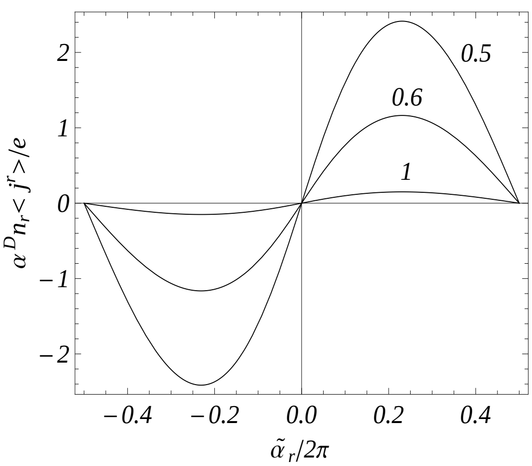

Figure 1: The quantity $\alpha^D n_r \langle j^r \rangle /e$ as a function of the phase in the quasiperiodicity condition for a conformally coupled scalar field in the $D = 4$ model with a single compact dimension. The graphs are plotted for $\alpha m = 0.25$ and the numbers near the curves are the corresponding values for the ratio L_r/η.

For the same model $D = 4$, $p = 3$, in figure 2, we have plotted the quantity $\alpha^D n_r \langle j^r \rangle$ for a conformally coupled scalar field versus the ratio L_r/η. The latter is the proper length of the compact dimension measured in units of the dS curvature scale α. In numerical evaluations we have taken $\tilde{\alpha}_r = \pi/2$ and the numbers near the curves correspond to the values of $m\alpha$. For small values of L_r/η, from the asymptotic analysis given above it follows that $n_r \langle j^r \rangle \propto (L_r/\eta)^{-D}$. As it has been explained before, depending on the parameter ν, for large values of L_r/η two different regimes arise with monotonic (for positive values of ν) and oscillatory (for imaginary values of ν) damping of the current density. Note that, for a conformally coupled field, ν is real for $m\alpha = 0.25$ and imaginary for $m\alpha = 2, 3$. In order to display the oscillatory behavior of the damping, on the right panel of figure 2 we have plotted the graph for $m\alpha = 3$. The value of the ratio L_r/η corresponding to the first zero of the current density decreases with increasing the value of $m\alpha$. For the first two zeros of the current density one has $L_r/\eta = 3.88, 8.29$ and $L_r/\eta = 2.87, 4.45$ in the cases $m\alpha = 2$ and $m\alpha = 3$, respectively.

4 Fermionic current

In this section we consider the VEV of the current density for a fermionic field $\psi(x)$ on the background of $(D + 1)$-dimensional dS spacetime with spatial topology $R^p \times (S^1)^q$. As before, the line-element is taken in the form (1). Assuming the presence of a classical gauge field A_μ, the dynamics of the Dirac spinor field is governed by the equation

$$i\gamma^\mu D_\mu \psi - m\psi = 0 \ , \ D_\mu = \partial_\mu + \Gamma_\mu + ieA_\mu. \tag{39}$$

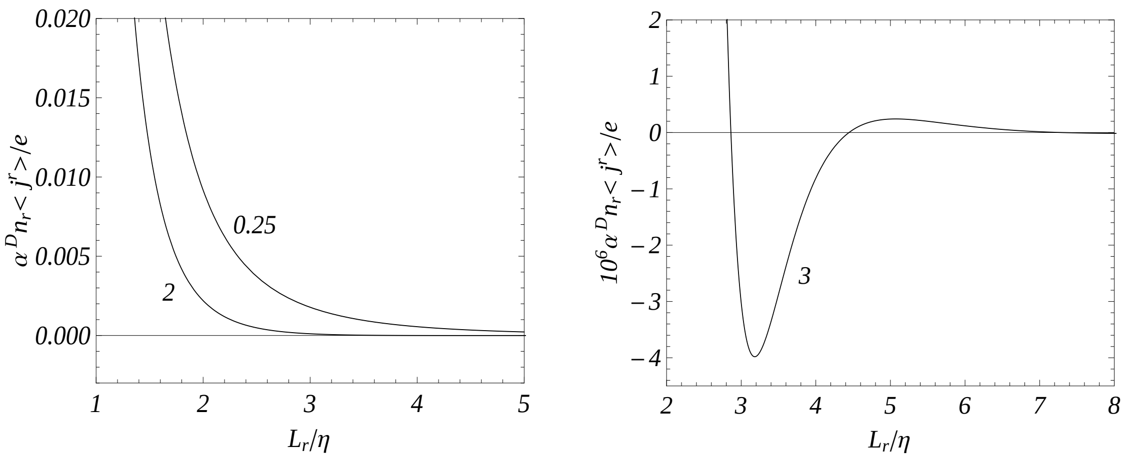

Figure 2: The current density for a conformally coupled scalar field in the model $D = 4$ and $p = 3$ as a function of the proper length of the compact dimension measured in units of the curvature scale α. The numbers near the curves are the values of the parameter $m\alpha$, and for the phase in the periodicity condition we have taken $\tilde{\alpha}_r = \pi/2$.

The Dirac matrices and the spin connection Γ_μ are expressed in terms of the flat-space Dirac matrices $\gamma^{(a)}$ by the relations

$$\gamma^\mu = e^\mu_{(a)}\gamma^{(a)},\ \Gamma_\mu = \frac{1}{4}\gamma^{(a)}\gamma^{(b)}e^\nu_{(a)}e_{(b)\nu;\mu}\ , \tag{40}$$

where the semicolon means the covariant derivative of vector fields and $e^\mu_{(a)}$ are the tetrads defined by $e^\mu_{(a)}e^\nu_{(b)}\eta^{ab} = g^{\mu\nu}$. For the background under consideration the tetrads can be taken in the form

$$e^{(0)}_\mu = \delta^0_\mu,\ e^{(a)}_\mu = e^{t/\alpha}\delta^a_\mu,\ a = 1, 2, \ldots, D. \tag{41}$$

In a $(D+1)$-dimensional spacetime the Dirac matrices are $N \times N$ matrices with $N = 2^{[(D+1)/2]}$, where the square brackets denote the integer part of the enclosed expression. Here we take the flat spacetime matrices in the Dirac representation

$$\gamma^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \gamma^{(a)} = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a^+ & 0 \end{pmatrix}. \tag{42}$$

with $a = 1, 2, \ldots, D$.

We need also to specify the periodicity conditions for the fermionic field along compact dimensions. Here these conditions will be taken in the form

$$\psi(t, \mathbf{x}_p, \mathbf{x}_q + L_l\mathbf{e}_l) = e^{i\alpha_l}\psi(t, \mathbf{x}_p, \mathbf{x}_q), \tag{43}$$

with constant phases α_l. Although we have used the same notation for the phases in the boundary conditions for scalar and fermionic fields, of course, in general they can be different. The fermionic condensate and the VEV of the energy-momentum tensor in the special cases of untwisted ($\alpha_l = 0$) and twisted ($\alpha_l = \pi$) fermionic fields and in the absence of a gauge field have been investigated in [14, 15].

Assuming a constant gauge field A_μ, we can exclude it from the field equation by the gauge transformation $A'_\mu = A_\mu + \partial_\mu\omega$, $\psi'(x) = e^{-ie\omega}\psi(x)$ with the function ω defined in (4). For the new field we have the quasiperiodicity condition

$$\psi'(t, \mathbf{x}_p, \mathbf{x}_q + L_l\mathbf{e}_l) = e^{i\tilde{\alpha}_l}\psi'(t, \mathbf{x}_p, \mathbf{x}_q), \tag{44}$$

with the new phase defined by (6). Similarly to the case for a scalar field, we will work in terms of the gauge transformed field omitting the prime.

The VEV of the current density for the fermionic field can be expressed in terms of the two-point function $S_{rs}^{(1)}(x,x') = \langle 0|[\psi_r(x), \bar{\psi}_s(x')]|0\rangle$, where r and s are spinor indices. The expression for the VEV reads

$$\langle j^\mu(x)\rangle \equiv \langle 0|j^\mu(x)|0\rangle = -\frac{e}{2}\mathrm{Tr}(\gamma^\mu S^{(1)}(x,x)). \tag{45}$$

Expanding the field operator in terms of a complete set of solutions to the Dirac equation, $\{\psi_\sigma^{(\pm)}(x)\}$, the following mode-sum formula can be obtained:

$$\langle j^\mu\rangle = -\frac{e}{2}\sum_\sigma \sum_{s=\pm} s\bar{\psi}_\sigma^{(s)}(x)\gamma^\mu\psi_\sigma^{(s)}(x). \tag{46}$$

In order to regularize the divergent expression in the right-hand side we will assume the presence of a cutoff function without writing it explicitly. The special form of this function will not be important for the further discussion.

For the problem under consideration, the complete set of fermionic mode functions can be found in a way similar to that used in [15] for special cases of twisted and untwisted fields. These mode function are presented in the form

$$\begin{aligned}
\psi_\sigma^{(+)} &= A_\sigma^{(+)}\eta^{(D+1)/2}e^{i\mathbf{k}\cdot\mathbf{x}}\begin{pmatrix} H_{1/2-i\alpha m}^{(1)}(k\eta)w_\chi^{(+)} \\ -i(\mathbf{n}\cdot\boldsymbol{\sigma}^+)H_{-1/2-i\alpha m}^{(1)}(k\eta)w_\chi^{(+)} \end{pmatrix}, & (47)\\
\psi_\sigma^{(-)} &= A_\sigma^{(-)}\eta^{(D+1)/2}e^{i\mathbf{k}\cdot\mathbf{x}}\begin{pmatrix} -i(\mathbf{n}\cdot\boldsymbol{\sigma})H_{-1/2+i\alpha m}^{(2)}(k\eta)w_\chi^{(-)} \\ H_{1/2+i\alpha m}^{(2)}(k\eta)w_\chi^{(-)} \end{pmatrix}, & (48)
\end{aligned}$$

where $\mathbf{n} = \mathbf{k}/k$, $\boldsymbol{\sigma} = (\sigma_1,\ldots,\sigma_D)$, and $\sigma = (\mathbf{k},\chi)$. In (47) and (48), $w_\chi^{(\pm)}$, $\chi = 1,\ldots,N/2$, are one-column matrices having $N/2$ rows with the elements $w_{\chi l}^{(+)} = \delta_{l\chi}$ and $w_\chi^{(-)} = iw_\chi^{(+)}$. For the normalization coefficients one has

$$|A_\sigma^{(\pm)}|^2 = \frac{ke^{\pi\alpha m}}{2^{p+2}\pi^{p-1}V_q\alpha^D}. \tag{49}$$

Similarly to the case of a scalar field, the eigenvalues for the components of the momentum along compact dimensions are given by the expressions (14). Note that for a massless field we have the standard conformal relation $\psi_\beta^{(\pm)} = (\eta/\alpha)^{(D+1)/2}\psi_\beta^{(\mathrm{M})(\pm)}$ between the mode functions defining the Bunch-Davies vacuum in dS spacetime and the corresponding mode functions $\psi_\beta^{(\mathrm{M})(\pm)}$ in the Minkowski spacetime with spatial topology $R^p\times(S^1)^q$.

Substituting the mode-functions into the mode-sum formula (46), it can be seen that the VEVs of the charge density and of the components along uncompactified dimensions vanish: $\langle j^\mu\rangle = 0$ for $\mu = 0,1,\ldots,p$. For the component of the current density along the rth compact dimension we get the expression

$$\begin{aligned}
\langle j^r\rangle &= \frac{ie\pi^{1-p/2}\eta^{D+2}e^{\pi\alpha m}N}{2^{p+2}V_q\alpha^{D+1}\Gamma(p/2)}\int_0^\infty dk_p\, k_p^{p-1}\sum_{\mathbf{n}_q} k_r \\
&\quad\times[H_{-1/2-i\alpha m}^{(1)}(k\eta)H_{1/2+i\alpha m}^{(2)}(k\eta) - H_{1/2-i\alpha m}^{(1)}(k\eta)H_{-1/2+i\alpha m}^{(2)}(k\eta)], \qquad (50)
\end{aligned}$$

where

$$k^2 = k_p^2 + \sum_{l=p+1}^{D}(2\pi n_l + \tilde{\alpha}_l)^2/L_l^2. \tag{51}$$

Further evaluation of the current density is similar to that for the case of the scalar field. Applying the summation formula (19) to the series over n_r in (50), one finds the expression

$$\begin{aligned}\langle j^r\rangle &= -\frac{4e\eta^{D+2}L_r^{1-p}N}{(2\pi)^{(p+3)/2}\alpha^{D+1}V_q}\int_0^\infty dz\, z\,\mathrm{Re}\left\{\left[I_{-1/2-i\alpha m}(z\eta)+I_{1/2+i\alpha m}(z\eta)\right]K_{1/2+i\alpha m}(z\eta)\right\} \\ &\quad \times\sum_{n=1}^\infty \frac{\sin(n\tilde{\alpha}_r)}{n^p}\sum_{\mathbf{n}_{q-1}^{(r)}} f_{(p+1)/2}(nL_r\sqrt{z^2+u_r^2}), \qquad (52)\end{aligned}$$

where u_r is defined by (21). Similarly to the case of the scalar field, the fermionic current density $\langle j^r\rangle$ is an odd periodic function of the phase $\tilde{\alpha}_r$ with the period 2π and it is an even periodic function of the phases $\tilde{\alpha}_l$, $l\neq r$, with the same period. In the absence of the gauge field the current density vanishes for periodic and antiperiodic boundary conditions. Similarly to the scalar case, the quantity $n_r\langle j^r\rangle$, which describes the charge flux through the hypersurface $x^r=$ const, depends on η and on the lengths of compact dimensions in the form of the combination L_l/η. The latter is the proper length of the compact dimension measured in units of α.

For a massless fermionic field the modified Bessel functions are expressed in terms of the exponential function and, after the integration by using the formula (27), from (52) we obtain the expression

$$\langle j^r\rangle = -\frac{2e(\eta/\alpha)^{D+1}N}{(2\pi)^{p/2+1}V_qL_r^p}\sum_{n=1}^\infty\frac{\sin(n\tilde{\alpha}_r)}{n^{p+1}}\sum_{\mathbf{n}_{q-1}^{(r)}} f_{p/2+1}(nL_ru_r). \qquad (53)$$

In this case one has the following relation between the fermionic and scalar current densities:

$$\langle j^r\rangle_{\mathrm{ferm}} = -N\langle j^r\rangle_{\mathrm{sc}}/2. \qquad (54)$$

Here we assumed that the phases α_l are the same for both fields. Note that in (54), $\langle j^r\rangle_{\mathrm{sc}}$ is the current density for a complex scalar field which is equivalent to two real scalar fields.

The transition to the Minkowskian limit is similar to that for the case of a scalar field. In this limit $\alpha\to\infty$ for a fixed t and $\eta\approx\alpha-t$. The dominant contribution to the integral in (52) comes from the integration region $z>m$. In this region we have

$$\left[I_{-1/2-i\alpha m}(z\eta)+I_{1/2+i\alpha m}(z\eta)\right]K_{1/2+i\alpha m}(z\eta)\approx\frac{1}{\alpha\sqrt{z^2-m^2}}. \qquad (55)$$

After the integration with the help of (27), to the leading order we get the current density in the Minkowski spacetime $\langle j^r\rangle\approx\langle j^r\rangle^{(\mathrm{M})}$ with

$$\langle j^r\rangle^{(\mathrm{M})} = -\frac{2eNL_r^{-p}}{(2\pi)^{p/2+1}V_q}\sum_{n=1}^\infty\frac{\sin(n\tilde{\alpha}_r)}{n^{p+1}}\sum_{\mathbf{n}_{q-1}^{(r)}} f_{p/2+1}(nL_r\sqrt{u_r^2+m^2}). \qquad (56)$$

This result coincides with the formula derived in [25] (in the notations of this reference $\tilde{\alpha}_l\to -2\pi\tilde{\alpha}_l$ and in [25] $A_l=\mathbf{A}_l$ in the definition for $\tilde{\alpha}_l$). The reason for the sign difference of α_l in the expression of $\tilde{\alpha}_l$ in (6) and in the corresponding expression of Ref. [25] is that, in the latter reference, for the evaluation of the VEV the negative-energy modes have been used, with the eigenvalues $k_l^{(+)}=2\pi(n_l+\alpha_l)/L_l$ instead of $k_l^{(-)}=2\pi(n_l-\alpha_l)/L_l$. This means that, in fact, the formulas given in [25] are for the periodicity conditions (43) with α_l replaced by $-2\pi\alpha_l$ (see also the comment in [26]). Comparing (56) with (32), we see that the relation (54) between the fermionic and scalar current densities holds in Minkowski spacetime for massive fields as well.

Hence, in the supersymmetric models on the background of the Minkowski spacetime with equal number of scalar and fermionic degrees of freedom the total current density vanishes. Note that this is not the case for the currents in the background of dS spacetime.

Now let us consider the fermionic current density in the asymptotic regions of the ratio L_r/η. For $L_r/\eta \ll 1$ the dominant contribution to (52) comes from the region with large values of $z\eta$. By taking into account that for large x one has $\left[I_{-1/2-i\alpha m}(x) + I_{1/2+i\alpha m}(x)\right] K_{1/2+i\alpha m}(x) \approx 1/x$, we see that to the leading order $\langle j^r \rangle$ coincides with the corresponding result for a massless fermionic field $\langle j^r \rangle \approx (\eta/\alpha)^{D+1} \langle j^r \rangle^{(\mathrm{M})}$. In this region we have the relation (54) between the scalar and fermionic field currents.

For small values of the ratio η/L_r, the dominant contribution to the integral in (52) comes from the range with small values of $z\eta$. By making use of the expansions for the modified Bessel functions for small values of the arguments, to the leading order we find

$$\langle j^r \rangle \approx -\frac{N B_f e^{-(D+1)t/\alpha}}{2^{p/2}\pi^{(p+3)/2} V_q L_r^p} \cos[2mt + 2\alpha m \ln(L_r/\alpha) + \phi_f]. \tag{57}$$

Here, B_f and ϕ_f are defined by the relation

$$B_f e^{i\phi_f} = 2^{i\alpha m}\Gamma\left(1/2 + i\alpha m\right) \sum_{n=1}^{\infty} \frac{\sin(n\tilde{\alpha}_r)}{n^{p+1-2i\alpha m}} \sum_{\mathbf{n}_{q-1}^{(r)}} f_{p/2+1-i\alpha m}(nL_r u_r). \tag{58}$$

Hence, unlike the scalar case, the damping of the fermionic current density for large values of the L_r/η is always oscillatory. Another difference is that the oscillation frequency does not depend on the curvature scale of dS spacetime and is completely determined by the mass of the field quanta.

Figure 3 presents the quantity $\alpha^D n_r \langle j^r \rangle /e$ for a fermionic field versus the phase in the quasiperiodicity condition for the $D = 4$ model with a single compact dimension ($p = 3$). The graphs are plotted for $\alpha m = 0.25$ and the numbers near the curves are the values for the ratio L_r/η.

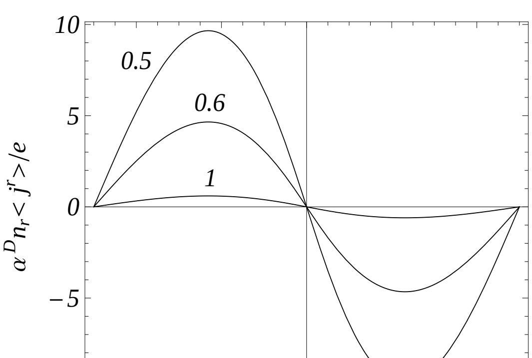

Figure 3: The current density as a function of the phase in the quasiperiodicity condition for a fermionic field in the $D = 4$ model with a single compact dimension. The graphs are plotted for $\alpha m = 0.25$ and the numbers near the curves are the corresponding values for the ratio L_r/η.

The dependence of the current density on the proper length of the compact dimensions is displayed in figure 4 for the same model $D = 4$, $p = 3$. The graphs are plotted for the phase

in the periodicity condition corresponding to $\tilde{\alpha}_r = \pi/2$ and the numbers near the curves are the values of the parameter $m\alpha$. For the fermionic field the damping of the current density for large values of L_r/η is always oscillatory. In order to display this behavior, on the right panel of figure 4 we present the current density for the case $m\alpha = 3$. As in the scalar case, the value of the ratio L_r/η for the the first zero of the current density decreases with increasing $m\alpha$. For the first two zeros we has $L_r/\eta = 3.04, 5.90$ and $L_r/\eta = 2.51, 3.67$ in the cases $m\alpha = 2$ and $m\alpha = 3$, respectively.

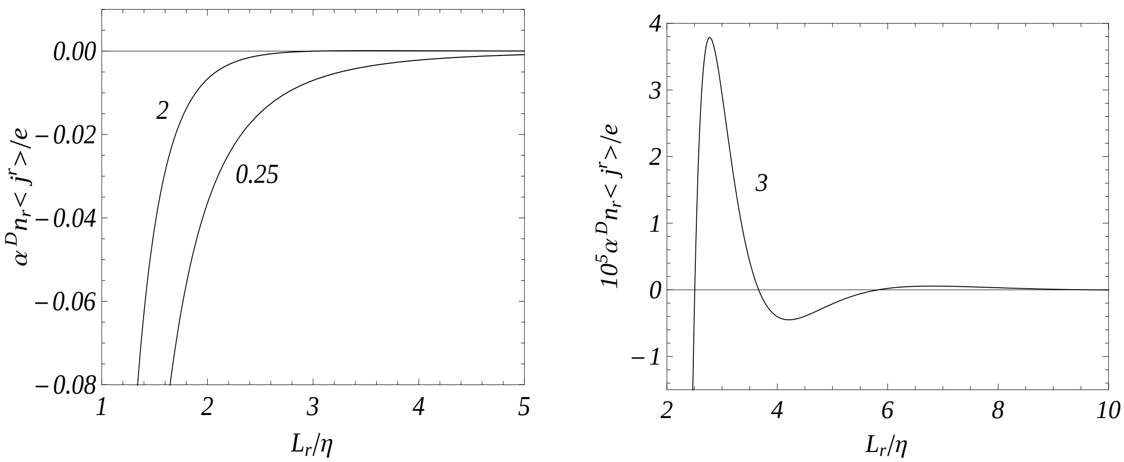

Figure 4: Fermionic current density as a function of the proper length of the compact dimension for separate values of the parameter $m\alpha$ (the numbers near the curves). The graphs are plotted for $\tilde{\alpha}_r = \pi/2$.

5 Conclusion

Among the most interesting consequences of the compactification of spatial dimensions is the appearance of nonzero expectation values for physical observables in the vacuum state of a quantum field. In the present paper we have considered the effect of the nontrivial topology on the VEVs of the current density for charged scalar and fermionic fields in the background of dS spacetime. In both cases we have assumed that the fields are prepared in the Bunch-Davies vacuum state. Along compact dimensions the quasiperiodicity conditions (3) and (43) are imposed with general phases. In addition, we have assumed the presence of a constant classical abelian gauge filed. This leads to the Aharonov-Bohm-like effect on the current density. The physical quantities depend on the phases in the periodicity conditions and on the gauge potential in the form of the combination (6). The VEVs of the charge density and of the components of the current density along compact dimensions vanish.

For the general case of toroidal spatial topology $R^p \times (S^1)^q$, the VEVs for the scalar and fermionic current densities along the rth compact dimension are given by the expressions (24) and (52), respectively. These VEVs are even periodic functions of the phases $\tilde{\alpha}_l$, $l \neq r$, with the period 2π, and they are odd periodic functions of the phase $\tilde{\alpha}_r$ with the same period. In particular, they exhibit Aharonov-Bohm-type oscillations as functions of the magnetic flux (with the period of the flux quantum) enclosed by the compact dimension. The current densities vanish in the special cases of untwisted and twisted fields, when the gauge field is absent. We have explicitly checked the transition to the Minkowskian results, previously discussed in [22, 25, 26], in the limit $\alpha \to \infty$, for a fixed value of t. In this limit, if the phases in the

periodicity conditions and the masses for scalar and fermionic fields are the same, we have the simple relation (54) between the scalar and fermioinic current densities. In particular, in supersymmetric models on the background of the Minkowski spacetime with equal number of scalar and fermionic degrees of freedom the total current density vanishes. The background gravitational field modifies the current densities for scalar and fermionic fields in different ways, and we have no similar cancellation in the dS spacetime.

If the proper length of the compact dimension is much smaller than the dS curvature scale, the effects induced by gravity are small and the current densities are related to the corresponding results for massless fields in Minkowski spacetime with compact dimensions by the simple formula $\langle j^r \rangle \approx (\eta/\alpha)^{D+1} \langle j^r \rangle^{(\mathrm{M})}$. For a fixed value of the ratio L_r/α, this limit corresponds to the early stages of the cosmological expansion ($t \to -\infty$), and the current densities behave as $e^{-(D+1)t/\alpha}$. The effect of gravity on the VEVs is decisive for proper lengths of the compact dimensions larger than the dS curvature radius. In this limit, for the case of a scalar field, depending on the mass, two regimes are realized. For the first one, corresponding to real values of the parameter ν defined by (12), the leading term in the asymptotic expansion of the current density is given by (34) and the VEVs decay monotonically as $(\eta/L_r)^{D-2\nu+2}$. For a fixed value of L_r/α, this corresponds to late stages of the cosmological evolution ($t \to +\infty$), and the current density is suppressed by the factor $e^{-(D-2\nu+2)t/\alpha}$. In the second regime, realized for imaginary values of ν, the behavior of the scalar current density is oscillatory, with the amplitude decaying as $(\eta/L_r)^{D+2}$. The period of oscillations is given by $\pi\alpha/|\nu|$. For the fermionic field the oscillatory regime is realized only (see (57)) with the amplitude decaying as $(\eta/L_r)^{D+1}$. In the fermionic case the period of oscillations does not depend on the curvature radius and is given by $2\pi/m$. Note that the decay rates for the amplitude in the cases of the scalar and fermionic currents are different. At late stages of the cosmological expansion, the current densities are suppressed by the factors $e^{-(D+2)t/\alpha}$ and $e^{-(D+1)t/\alpha}$ for scalar and fermionic fields, respectively. For the values of the curvature scale $\alpha \gtrsim m^{-1}$, if the length of the compact dimension is much larger than the Compton wavelength of the field quanta, $mL_r \gg 1$, the decay of the current densities as functions of mL_r is a power-law, for both scalar and fermionic fields. This is in contrast to the case of the Minkowski bulk, where the VEVs of the current densities are suppressed exponentially.

The effects we have discussed can be applied to two types of models. For the first one $D = 3$ and the results given here describe how the properties of the universe are changed by one-loop quantum effects induced by the compactness of spatial dimensions. Though the observational data constrain the size of possible compact dimensions to be larger than the horizon scale, at early stages of the cosmological expansion the lengths of compact dimensions were small and the effects considered here can be important. The second class of models, with $p = 3$ and $D > 3$, correspond to the universe with Kaluza-Klein-type extra dimensions. In these models, the current density along compact dimensions could be a source for cosmological magnetic fields. The gravitational analog of this electromagnetic effect was discussed in [11], where it was shown that the topological Casimir energy can be considered as a possible origin for the dark energy.

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